Water and Objects (10 pt)

In this problem, we consider phenomena caused by the interaction between water and objects, related to surface tension. Part A treats motion, while Parts B and C consider static situations.

If necessary, you can use the fact that if a function $y(x)$ satisfies the differential equation $y''(x) = ay(x)$ (a is a positive constant), then its general solution is $y(x) = Ae^{\sqrt{a}x} + Be^{-\sqrt{a}x}$, where A and B are arbitrary constants.

Part A. Merger of water drops (2.0 points)

As shown in Fig.1, we consider two stationary, spherical water drops on the surface of a super-hydrophobic material, meaning that a very strong repulsive force exists between the material and water.

Initially, two neighbouring, identical spherical water drops are placed on the surface. These two drops then merge after touching each other and form a larger spherical water drop, which suddenly jumps up.

A.1	The radius a of both water drops before the merger is $100\ \mu$m. The density of water ρ is 1.00×10^3 kg/m^3. The surface tension γ is 7.27×10^{-2} J/m^2. A fraction k of the difference between the surface energy before and after the merge, ΔE, is transformed into the kinetic energy of the ejected water drop. Then, determine the initial jump-up velocity, v, of the merged water drop to two significant figures under the following assumptions: • $k = 0.06$ • Before and after the merge, the total volume of water is conserved.	2.0pt

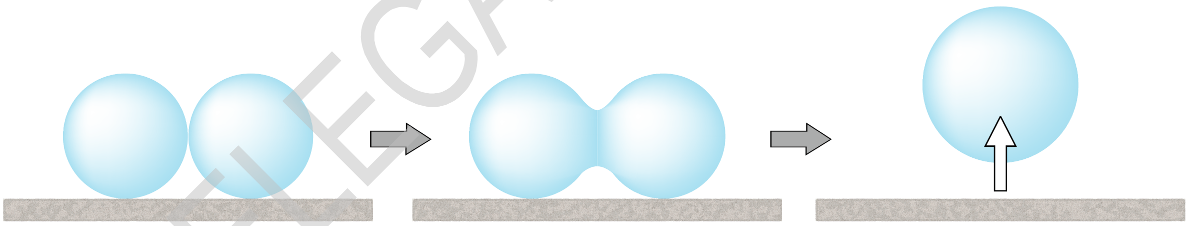

Fig. 1: Merging of two water drops and jump of the merged water drop.

Part B. A vertically placed board (4.5 points)

A flat board is immersed vertically in water. Figures 2(a) and 2(b) show water surface shapes for the hydrophilic (attractive) and hydrophobic board materials respectively. We neglect the thickness of the board.

The board surface is in the yz-plane, and the horizontal water surface far away from the board is in the xy-plane with $z = 0$. The surface shape does not depend on the y-coordinate. Let $\theta(x)$ be the angle between the water surface and the horizontal plane at a point (x, z) on the water surface in the xz-plane. Here $\theta(x)$ is measured with respect to the positive x-axis and the anticlockwise rotation is taken as positive. Let $\theta(x)$ be θ_0 at the point of contact between the board and the water surface ($x = 0$). In the following, θ_0 is fixed by the properties of the board material.

The water density ρ is constant and the water surface tension γ is uniform. The gravitational acceleration constant is given by g. The atmospheric pressure, P_0, is assumed to be always uniform. Let us determine the water surface shape in the following steps. Note that the unit of surface tension is J/m^{2}or equivalently N/m.

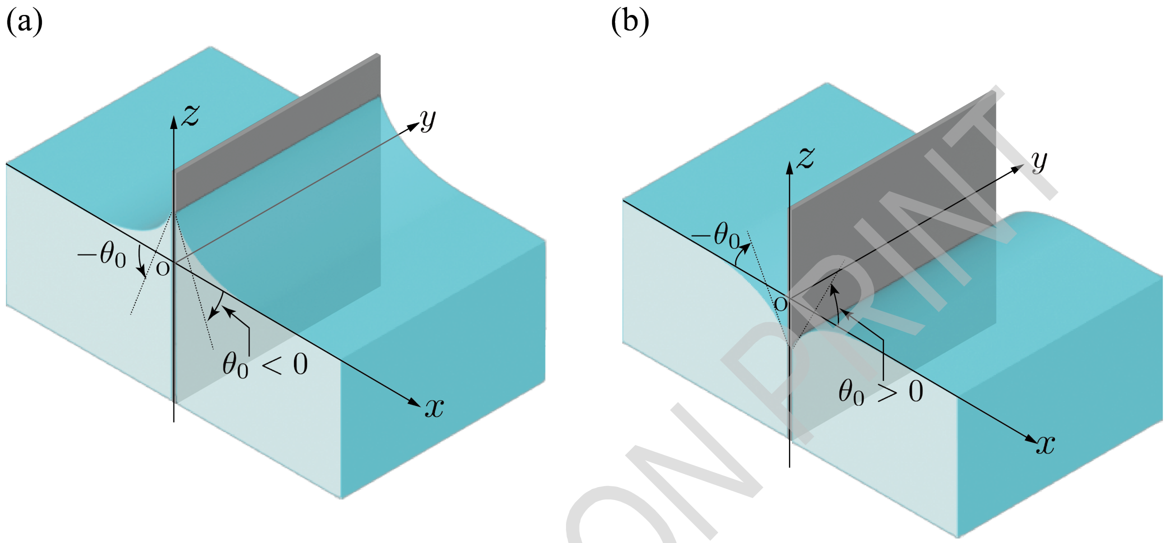

Fig. 2: Boards vertically immersed in the water. (a) hydrophilic board case; (b) hydrophobic board case.

B.1	We consider the case of a hydrophilic board, as shown in Fig.2(a). We note that the water pressure, P, satisfies the conditions: $P < P_0$ for $z > 0$ and $P = P_0$ for $z = 0$. Express P at z in terms of ρ, g, z, and P_0.	0.6pt

B.2	We consider a water block whose cutout is shown as shaded in Fig.3(a). Its xz-plane cross-section is shown in the hatched area in Fig.3(b). Let z_1 and z_2 be the left and right edge coordinates, respectively, of the boundary (water surface) between the water block and the air. Obtain the horizontal component (x-component) of the net force per unit length along the y-axis, f_x, which is exerted on the water block due to the pressure, in terms of ρ, g, z_1, and z_2. Note that P_0 results in no net horizontal force on the water block.	0.8pt

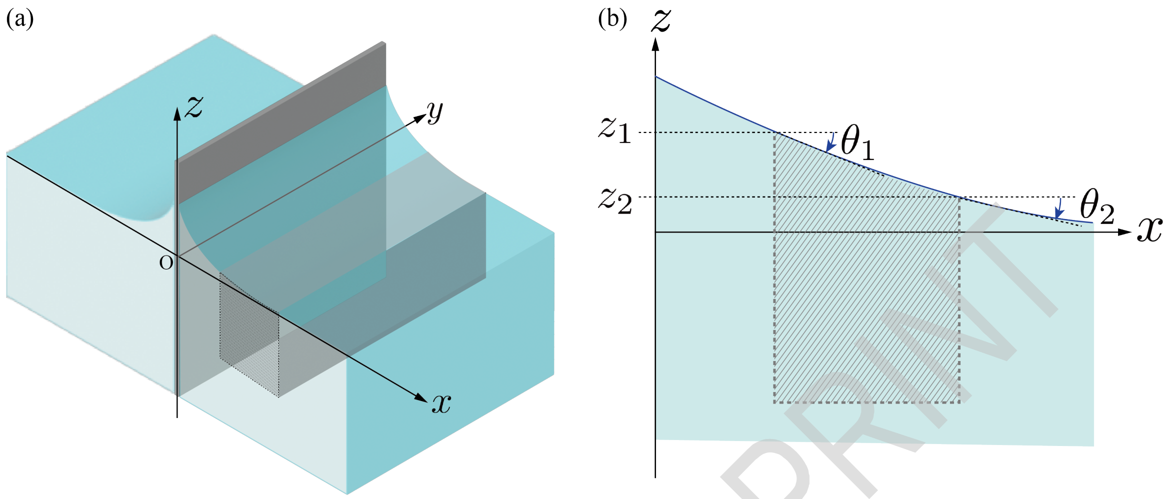

Fig. 3: Cutout form of the water block on the water surface. (a) Bird's eye view and (b) cross-sectional view.

B.3	Surface tension acting on the water block is balanced with the force f_x discussed in B.2. We define θ_1 and θ_2 as the angles between the water surface and the horizontal plane at the left and right edges respectively. Express f_x in terms of γ, θ_1, and θ_2.	0.8pt

B.4	The following equation holds at an arbitrary point (x, z) on the water surface, $$\frac{1}{2}\left(\frac{z}{\ell}\right)^a + \cos\theta(x) = \text{constant}. \quad (1)$$ Determine the exponent a and express the constant ℓ in terms of γ and ρ. Note that this equation holds regardless of hydrophilic or hydrophobic board materials.	0.8pt

B.5	In Eq. (1) in B.4, we assume that variation of the water surface is slow, i.e., $\lvert z'(x)\rvert \ll 1$, so that we can expand $\cos\theta(x)$ with respect to $z'(x)$ up to the second order. Then, differentiating the resultant equation with respect to x, we obtain the differential equation satisfied by $z(x)$. Solve this differential equation and determine $z(x)$ for $x \geq 0$ in terms of $\tan\theta_0$ and ℓ. Note that the vertical directions of Figs. 2 and 3 are exaggerated and they do not satisfy the condition, $\lvert z'(x)\rvert \ll 1$.	1.5pt

Part C. Interaction between two rods (3.5 points)

Identical rods A and B made of the same material are floating in parallel on the water surface and are placed at the same distance away from the y-axis (Fig.4).

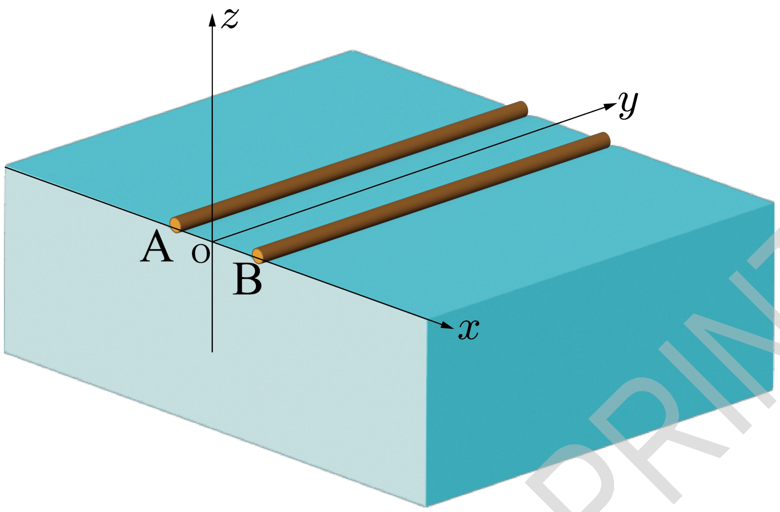

Fig. 4: Two rods A and B floating on the water surface.

C.1	At the contact points of the rod B and the water surface, we define the z-coordinates z_a and z_b, and the angles θ_a and θ_b, as shown in Fig.5. Determine F_x, the horizontal component of force on the rod B per unit length along the y-axis , in terms of θ_a, θ_b, z_a, z_b, ρ, g, and γ.	1.0pt

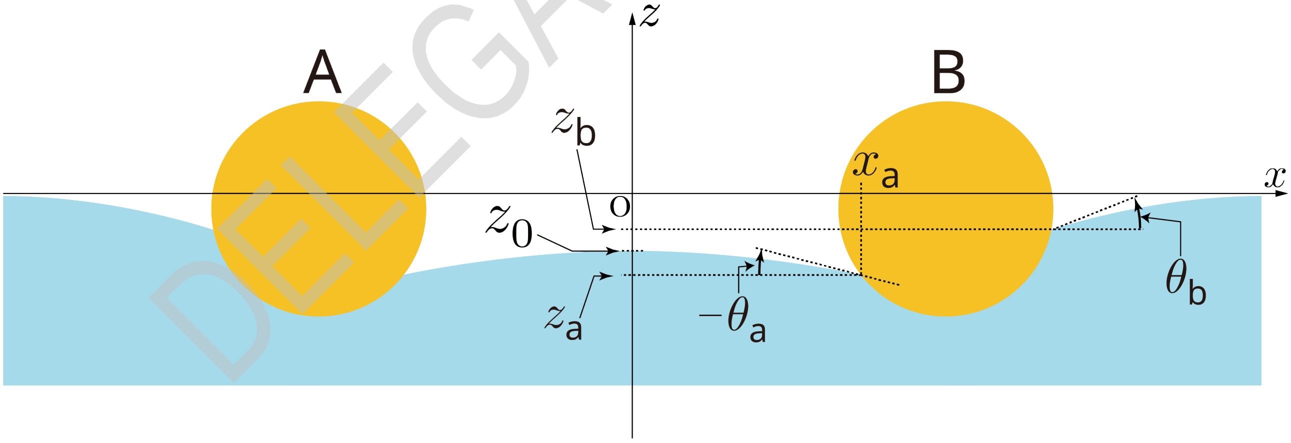

Fig. 5: Vertical cross-sectional view of two rods floating on the water surface.

C.2	We define the z-coordinate of the water surface, z_0, at the midpoint of two rods in the xz-plane. Express the force F_x obtained in C.1 without using θ_a, θ_b, z_a, and z_b.	1.5pt

C.3	Let x_a be the x-coordinate of the contact point between the water surface and the left side of the rod B. Using the differential equation obtained in B.4, express the water level coordinate z_0 of the midpoint of these two rods A and B in terms of x_a and z_a. You can use the constant ℓ introduced in B.4.	1.0pt